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# PREDICTING THE BENEFITS OF TOPOLOGY OPTIMIZATION

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## ABSTRACT*

Topology optimization is a systematic method of generating designs that maximize specific objectives. While it offers significant benefits over traditional shape optimization, topology optimization can be computationally demanding and laborious. Even a simple 3D compliance optimization can take several hours. Further, the optimized topology must typically be manually interpreted and translated into a CAD-friendly and manufacturing friendly design.

This poses a predicament: *given an initial design, should one optimize its topology*? In this paper, we propose a simple metric for predicting the benefits of topology optimization. The metric is derived by exploiting the concept of topological sensitivity, and is computed via a finite element swapping method. The efficacy of the metric is illustrated through numerical examples.

## INTRODUCTION

Design is an iterative process; with the advent of advanced computing methods, various strategies have been proposed to reduce design cycles. Topology optimization [1] is one such method to construct and discover novel designs. In topology optimization, one starts with an initial design, on which a structural problem is posed; see Figure 1.

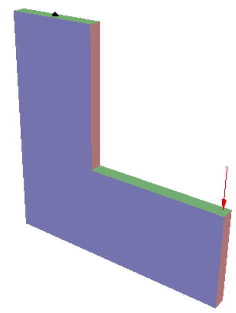


Figure 1: A STRUCTURAL PROBLEM OVER A DESIGN-SPACE.

In this example, it is assumed that the initial design coincides with the allowable design space, but this need not be the case. Then, using finite element analysis (FEA), and one of the various topology optimization methods such as SIMP [2]–[5], evolutionary [6]–[8], or level-set [9]–[11], an optimal topology is constructed.

For the problem posed in Figure 1, if the objective is compliance, the optimal topology for a volume fraction of 0.5, in the absence of other constraints, is illustrated in Figure 2(A). On the other hand, if the objective is the p-norm von Mises stress [12], an optimal topology is illustrated in Figure 2(B). Such insights can be particularly valuable during the initial stages of design.

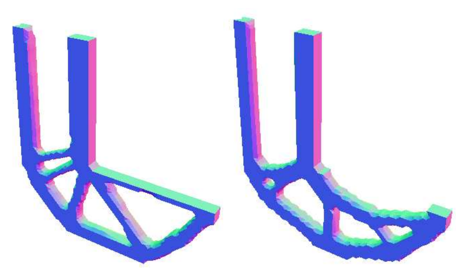


Figure 2: TOPOLOGIES THAT MINIMIZE: (A) COMPLIANCE, (B) STRESS.

Topology optimization has been used to design aircraft components [13], [14], spacecraft modules [15], automobiles components [16], cast components [17], compliant mechanisms [18]–[21], etc.

Unfortunately, it can be a computationally demanding task. For example, even a simple compliance minimization problem in 3D can take several hours for completion [22], while stress minimization and imposition of manufacturing constraints can take several days for completion [23]. It can also be laborious in that the optimized topology must often be interpreted and converted into a CAD-friendly and/or manufacturing-friendly parametric design.

Thus, while advanced computing methods such as topology optimization exists, the high computational and labor costs poses a predicament to the designer: *Given an initial design, such as the one in Figure 3, should one optimize its topology? Can one predict the potential benefits before embarking on a time-consuming process?*

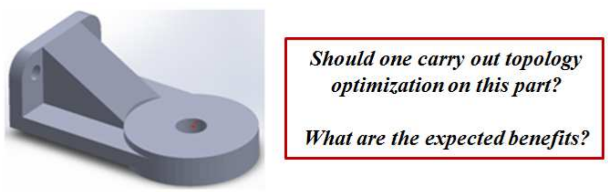



Figure 3: AN EXAMPLE TO ILLUSTRATE THE RESEARCH PROBLEM.

At first glance, it may appear that such questions cannot be answered without first carrying out a topology optimization study! However, in this paper we demonstrate that it is indeed possible to estimate the benefits through computationally efficient and robust algorithms, requiring little or no human input.

As designs grow in complexity, such value-driven questions will become even more important. For example, given an assembly of parts (see Figure 4), *which of the parts, if any, should one optimize? How do we rank-order these parts for optimization?*


Figure 4: EXTENSION OF RESEARCH QUESTION TO ASSEMBLIES.

In this paper, we propose a simple metric, based on the concept of *topological sensitivity* [24]–[28], for predicting the benefits of topology optimization.

# LITERATURE REVIEW

## Topology Optimization Methods

Broadly, there are three popular classes of topology optimization methods today: Solid Isotropic Material with Penalization (SIMP), level-set and evolutionary.

Among these, SIMP is perhaps the most widely used [29]. In the popular finite element formulation of SIMP, a density variable is assigned to each element [2], [30] and optimized (see Figure 5); typical optimization in SIMP can take 100's of finite element operations. Most commercial topology optimization systems such as Optistruct [31], Genesis [32], and Atom [33] are based on SIMP. The primary advantages of SIMP are that it is easy to implement and the theoretical foundation is well established. However, the ill-conditioning of the stiffness matrices [34], due to presence of low-density elements, can lead to high computational costs for iterative solvers [22], [23] in 3D. The strengths and weaknesses of SIMP are inherited by commercial implementations.

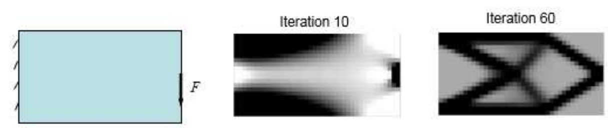



Figure 5: A TYPICAL STRUCTURAL PROBLEM, AND PROGRESSION IN SIMP.

The second class of topology optimization methods define the evolving topology via a level-set function that is typically controlled via Hamilton-Jacobi equations [35]. An important advantage of level-set methods over SIMP is the unambiguous description of the boundary. Consequently, level-set based methods are particularly effective in boundary-dependent problems and stress-constrained topology optimization. Numerous authors have demonstrated the success of level-set methods; for example, see [36], [37], [38].

The third class are the evolutionary methods; among these, bi-directional evolutionary structural optimization (BESO) [39], is the most popular. BESO starts from an initial design space, and iterates to the final topology by removing 'undesirable' elements, and simultaneously adding 'desirable' elements. It is argued in [40] that BESO can search the entire design domain more thoroughly compared with traditional methods, with a better likelihood of finding the global optimum. However, BESO also suffers from several critical shortcomings as pointed out in [41].

## Computational Challenges

Since all topology optimization methods entail repeated finite element analysis, the computational cost is high, independent of the method (some being more expensive than the others). For example, in [42], problems with 1~3 million degrees of freedom were optimized in 3~40 hours (depending on the specific problem) on a Cray T3E super computer. In [22], using specialized Krylov recycling methods, problems with about 1 million degrees of freedom was optimized in 45 hours on a regular desktop. Using Optistruct (2013 release) [31], the benchmark problem posed in [22] was solved in 20 hours on a high-end server.

All of the above problems are simple single-load unconstrained compliance-minimization problems. The challenges increase many-fold in non-compliance and multi-load problems.

One possible strategy of reducing the computational effort is to use a coarse finite element mesh, but this is not desirable for at least two reasons: (1) coarse-meshes do not accurately capture the behavior of a structure, leading to erroneous results during optimization, and (2) disconnected topologies are more likely to occur with a coarse mesh.

In summary, while numerous advances have been made, the current challenges in topology optimization beg strategic questions: *Given an initial design, is it worth carrying out topology optimization? Can one estimate the potential benefits, prior to optimizing?*

## TECHNICAL BACKGROUND

In this Section, we define a quantifiable metric for predicting potential benefits of topology optimization. The metric exploits the concept of topological sensitivity discussed next.

### Topological Sensitivity

The proposed methodology rests on the concept of topological sensitivity that captures the first order impact of inserting a small circular hole within a domain on various quantities of interest. This concept has its roots in the influential paper by Eschenauer [43], and has later been extended and explored by numerous authors [24]–[28], including generalization to arbitrary features [44]–[46].

To illustrate the topological sensitivity concept, consider the design illustrated earlier in Figure 5a. Consider now inserting a small hole within the domain, i.e., modifying the topology, as in Figure 6a. Clearly, the structural response will change, and so will various quantities of interest. Topological sensitivity (TS) is the expected change in a quantity of interest $q$ due to an infinitesimal topological change at a particular location $p$. If the quantity of interest is the compliance, one can show that the desired sensitivity in 2-D is given by [47]:

$$\mathcal{T}(p) \equiv \lim_{\varepsilon \to 0} \frac{q(p;\varepsilon) - q}{\pi \varepsilon^2} = \frac{4}{1+\nu} \sigma : \varepsilon - \frac{1-3\nu}{1-\nu^2} tr(\sigma) tr(\varepsilon) \quad (1)$$

TS is a spatial field in that the sensitivity depends on where the hypothetical hole is inserted. Similar expressions can be deduced in 3-D, and for various quantities of interest [44]. The TS field for the problem posed in Figure 5(A) is illustrated in Figure 6(B).

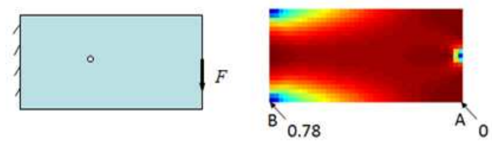



Figure 6: (A) TOPOLOGICAL CHANGE, (B) TOPOLOGICAL SENSITIVITY FIELD.

While the TS field has been used for optimization [23], it is not the main focus of this paper. The objective here is to use TS as a means of estimating the benefits of topology optimization.

*Observe that regions with low TS (for example, near point-A) are less critical than regions with high TS (for example, near point-B). Further, the TS carries quantitative information on how the quantity of interest will change if the domain is modified; it follows from above that:*

$$q(p;\varepsilon) \approx q + \pi \varepsilon^2 \mathcal{T}(p) \quad (2)$$

*The proposed metric and algorithm discussed in the next Section rest on this simple observation.*

## PROPOSED METHOD

In this Section, we present the proposed metric and algorithm. The proposed method relies on the topological sensitivity concept, and also borrows ideas from the BESO method [39].

### Proposed Metric

Consider the 2-D design illustrated in Figure 7, subject to a structural load. Observe that initial design is a proper subset of the allowable space. In other words, one can subtract material from the initial design or add material to it within the allowable space. For simplicity, we shall assume that the designer is interested in two conflicting quantities of interest: compliance (J) and the volume (V).

*Should the initial design be optimized within the allowable space? Are there significantly better designs within the allowable space, for example, with the same volume, but lower compliance, or same compliance but lower volume?*

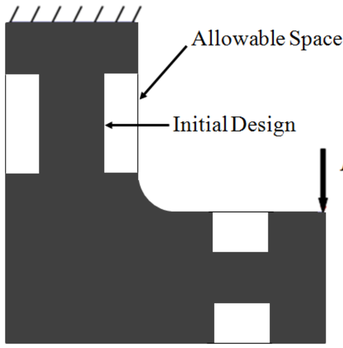



Figure 7: THE INITIAL DESIGN AND ALLOWABLE SPACE.

To answer this question, observe that the initial design can be represented as a point $(V_0, J_0)$ in the volume-compliance graph as in Figure 8.

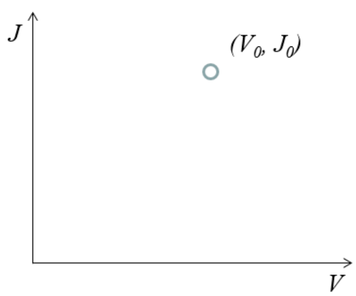



*Figure 8:* THE INITIAL DESIGN POINT.

Now consider the *hypothetical* problem of minimizing compliance while keeping volume a constant; this is illustrated schematically in Figure 9. It is well known that there exists an optimal solution for the compliance minimization problem [48]. Of course, the optimal solution, and the possible reduction in compliance $\Delta J$ are unknown.

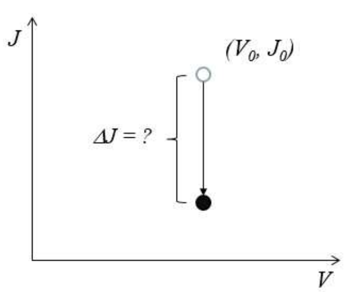



Figure 9: MINIMIZATION OF COMPLIANCE.

Similarly, consider the *hypothetical* minimization of volume while keeping compliance a constant (see Figure 10). Once again, the final topology and volume reduction $\Delta V$ are unknown.

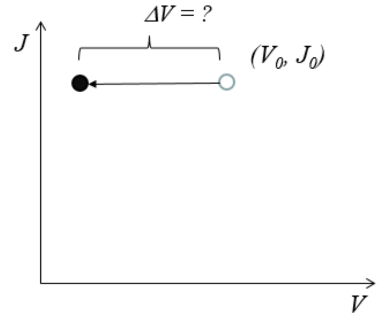



Figure 10: MINIMIZATION OF VOLUME.

To estimate the benefits of topology optimization, it is sufficient to *estimate* $\Delta J$ and $\Delta V$. Once these two quantities are estimated (see next few sections), one can compute the normalized distances $(\Delta V / V_0, \Delta J / J_0)$ that range from 0 to 1. These normalized distances are the proposed metrics for a given design.

If a particular normalized distance is close to zero, it implies that the design is close to being optimal, i.e., topology optimization is unlikely to yield a significant reduction in the quantity of interest (along that direction). On the other hand, if a metric is close to 1, then the design point is far from being optimal, suggesting significant benefits from topology optimization. Examples provided in the next few Sections support this argument. The cutoff value may be case-dependent, and requires further investigation.

## Metric Estimation

Given a design in Figure 7, we shall assume an FEA has been carried out, and the topological sensitivity for compliance has been computed. A finite element mesh for the design is illustrated in Figure 11.

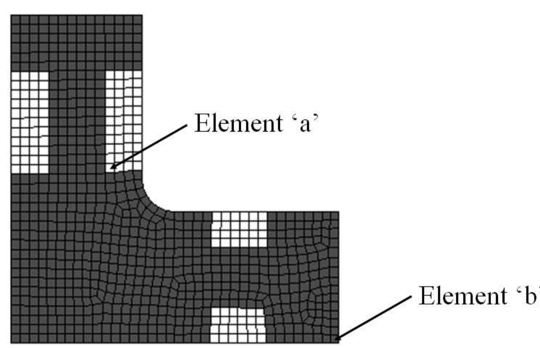



Figure 11: THE INITIAL DESIGN AND ALLOWABLE SPACE ARE MESHED WITH FINITE ELEMENTS.

Consider two elements 'a' and 'b' identified in Figure 11, where element 'a' is outside the initial design, while element-b is inside the design; we will assume that the two elements are approximately of equal area. Let the topological sensitivity (TS) of element 'a' be 0.9 while that of element 'b' be 0.1.

Recall from Section 3.1 that TS values in finite element mesh indicate how important a specific element is for the objective of interest. Since TS value of element 'a' is higher than that of element 'b', element 'a' is significantly more important to the structure than element 'b'. Therefore, to minimize compliance, while keeping volume a constant, one can insert element 'a' and delete element 'b' as illustrated in Figure 12.

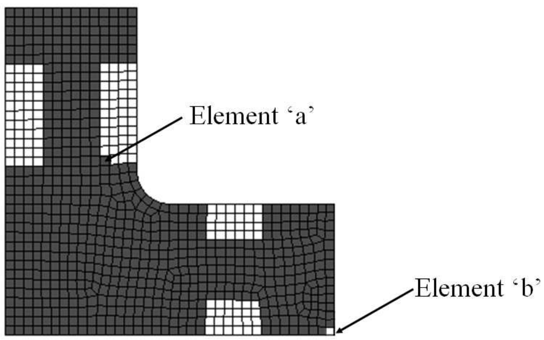



Figure 12: THE DESIGN AND ALLOWABLE SPACE AFTER SWAPPING A SINGLE PAIR OF ELEMENTS.

The change in compliance can be computed via Eqn. (2) , and one can now repeat the process, leading to element-swapping algorithm discussed next.

### Element-Swapping Algorithm for Estimating $\Delta J$

The following element-swapping algorithm estimates the possible reduction in compliance $\Delta J$ , while keeping volume (approximately) a constant.

1. Set the estimate $\Delta J = 0$
2. Carry out a finite element study on the given design
3. Compute the topological sensitivity (TS) for compliance
4. Sort all 'out' elements in a decreasing order of TS values
5. Sort all 'in' elements in an increasing order of TS values
6. Pick the first element 'a' from the 'out' list, and the first element 'b' from the 'in' list.
7. If the TS-field at 'b' is less than the TS-field at 'a', then:
8. Swap(a, b), i.e., insert element 'a', and delete element 'b'; remove these two elements from their respective lists.
9. Update:
   $\Delta J = \Delta J + TS(a) * Volume(a) - TS(b) * Volume(b)$
   Go back to step-6
10. Else:
    Stop.

### Illustrative Example

If one executes the above algorithm on the design in Figure 11, the predicted topology and the estimated normalized distance ($\Delta J / J_0 = 0.76$) are illustrated in Figure 13(A). In comparison, if one did carry out a full topology optimization study (requiring numerous FEAs), Figure 13(B) illustrates the optimized topology and the actual normalized distance ($\Delta J / J_0 = 0.83$).

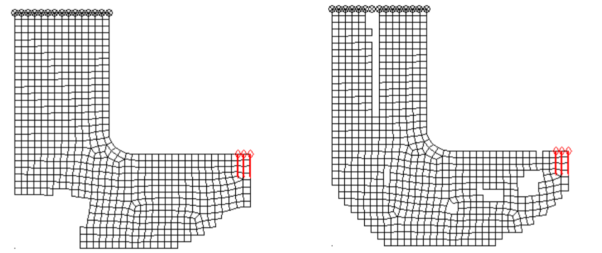


Figure 13: (A) PREDICTED TOPOLOGY BY 1FEA, AND (B) ACTUAL TOPOLOGY REQUIRED BY 70-FEA.

Observe the following:

1. Although the predicted topology differs significantly from the optimized topology, the estimated normalized distance suggests that the initial design in Figure 7 is far from optimal, and would benefit from topology optimization.
2. The swapping algorithm relies on a single finite element analysis, while topology optimization entails numerous (~70) finite element analysis and other sensitivity calculations.
3. The accuracy of the estimation can be further improved by using multiple FEAs. Specifically, in step 9, instead of returning back to step-6, one can (optionally) return back to step-2. This leads to an update in the TS values; for example, Figure 14 illustrates the improved topology and metric ($\Delta J / J_0 = 0.78$) if one allows for 5 FEAs.

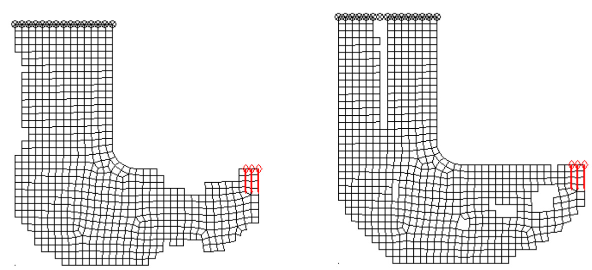


Figure 14: (A) PREDICTED TOPOLOGY BY 5FEA, AND (B) ACTUAL TOPOLOGY.

### Element-Swapping Algorithm for Estimating $\Delta V$

Similarly, to estimate $\Delta V$ (see Figure 10), the swapping algorithm is modified as follows.

1. Set the estimate $\Delta V = 0$
2. Carry out a finite element study on the given design
3. Compute the topological sensitivity (TS) field
4. Sort all 'in' elements in an increasing order of TS values

5. Sort all 'out' elements, in a decreasing order of TS values
6. Pick the first element 'a' from the sorted list in step-5, Find the set of elements '$b_i$' from the sorted list in step-4 such that the sum(Volume($b_i$)*TS($b_i$)) is greater than or equal to Volume(a)*TS(a).
   If the set is null
      Stop
   Else
      Delete all elements $b_i$, and insert element-a; remove these elements from their respective lists.
      Update $\Delta V$ as follows, and go back to step 6:
      $\Delta V = \Delta V + Volume(a) - \sum_i Volume(b_i)$

Observe that, in order to reduce volume, far more elements are deleted than inserted, while the compliance remains (nearly) a constant.

## Illustrative Example

If one executes the above algorithm on the design in Figure 7, the final topology ( $v = 0.72$ ) and the estimated normalized metric ( $\Delta V / V_0 = 0.13$ ) are illustrated in Figure 15(A), while the optimized topology ( $v = 0.35$ ) and actual metric ( $\Delta V / V_0 = 0.57$ ) are illustrated in Figure 15(B).

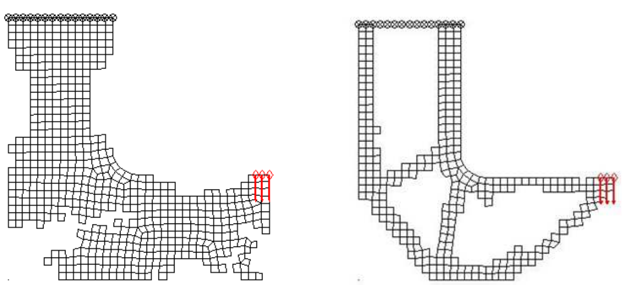


Figure 15: (A) 1 FEA BASED PREDICTED TOPOLOGY AND (B) ACTUAL TOPOLOGY BY 43-FEA

As before, additional FEAs can improve the accuracy. For example, Figure 16(A) shows the final topology ( $v = 0.53$ ) and estimated normalized metric ( $\Delta V / V_0 = 0.36$ ) when 5 FEAs are permitted.

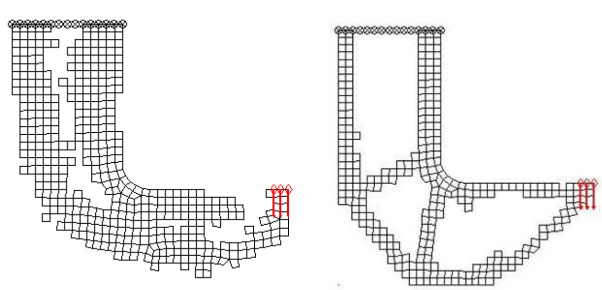


Figure 16: 5 FEA BASED PREDICTED RESULTS VERSUS ACTUAL TOPOLOGY

It can be observed that the proposed element swapping method shares some similarities with evolutionary structural optimization (ESO) [39]. However, the proposed method is notably different from ESO in that the proposed method depends on mathematically-rigorous topological sensitivity that can be generalized to any quantity of interest. On the other hand, ESO exploits quantities such as von Mises stress and strain energy density [41] that are at best, applicable to compliance minimization problems.

# 3D NUMERICAL EXPERIMENTS

In this Section, we demonstrate the efficacy of the proposed method through numerical experiments in 3D. The default material properties are $E = 2*10^{11}$ Pa and $v = 0.33$.

## Flange Problem

The first experiment involves the flange with the allowable space illustrated in Figure 17.

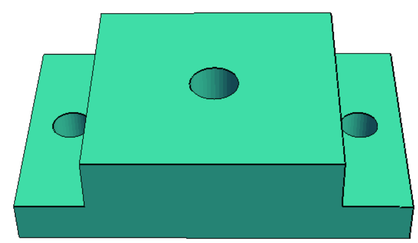


Figure 17: ALLOWABLE SPACE FOR THE FLANGE PROBLEM.

An initial design is illustrated in Figure 18; it is fixed on the two side-holes, while a unit vertical load is applied in the middle hole. For FEA, the structure is discretized into 10,000 elements.

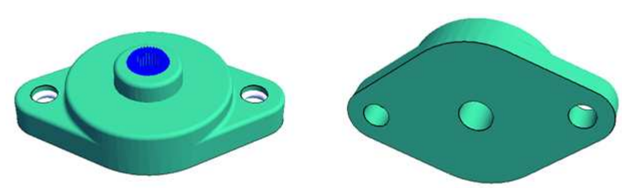


Figure 18: INITIAL DESIGN FOR THE FLANGE (TWO VIEWS).

The estimation for the $\Delta J/J_0$ and $\Delta V/V_0$ are carried out using both 1-step FEA and 5-step FEA approximations, and the results are summarized in Table 1.

As expected, the 5-FEA predictions are more accurate than the 1-FEA predictions. Since the predicted metrics are small, one can conclude that the initial design in Figure 18 is not worth optimizing. This is consistent with the actual metrics in Table 1.

Table 1: ESTIMATION RESULTS FOR INITIAL DESIGN IN Figure 18.

| | 1 FEA | 5 FEA | Actual (~70 FEAs) |
|---|---|---|---|
| $\Delta J/J_o$ | 0.04 | 0.05 | 0.07 |
| $\Delta V/V_o$ | 0.31 | 0.12 | 0.08 |

### Thick Plate

As a second example, consider the 3D thick plate in Figure 19 that will serve as the allowable space.

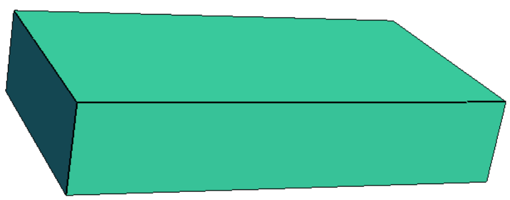


Figure 19: ALLOWABLE SPACE FOR THE THICK PLATE PROBLEM.

An initial design is illustrated in Figure 20; it is fixed at both sides and a uniformly distributed load is applied on the top surface. The structure is meshed with 12,000 finite elements.

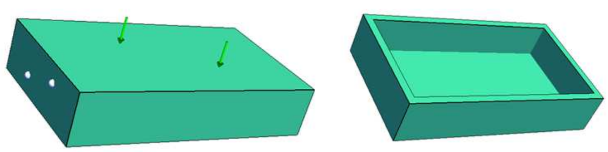


Figure 20: INITIAL DESIGN (TWO VIEWS).

The estimation for the $\Delta J/J_o$ and $\Delta V/V_o$ using 1-step FEA and 5-step FEA approximations, and the results are summarized in Table 2. Observe that the predicted metrics are reasonably high, suggesting that the initial design in Figure 20 is far from optimal. This is consistent with the actual improvement in the metrics (see Table 2).

Table 2: ESTIMATION RESULTS FOR INITIAL DESIGN IN Figure 19.

| | 1 FEA | 5 FEA | Actual (~70 FEAs) |
|---|---|---|---|
| $\Delta J/J_o$ | 0.24 | 0.44 | 0.59 |
| $\Delta V/V_o$ | 0.52 | 0.28 | 0.31 |

### Knuckle Problem

The third experiment involves the 3-D knuckle illustrated in Figure 21, that will serve as the allowable space. The structure is fixed at both bottom holes and a force is applied on the top hole. For FEA, the domain was discretized into 13,000 elements.

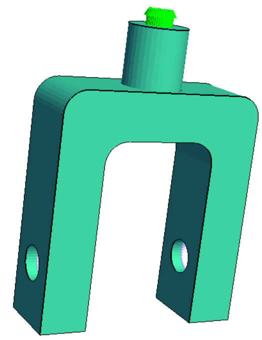


Figure 21: ALLOWABLE SPACE AND THE STRUCTURAL PROBLEM.

This time, we will consider eight different initial designs are in Figure 22; the objective is to estimate the metrics for each of these designs.

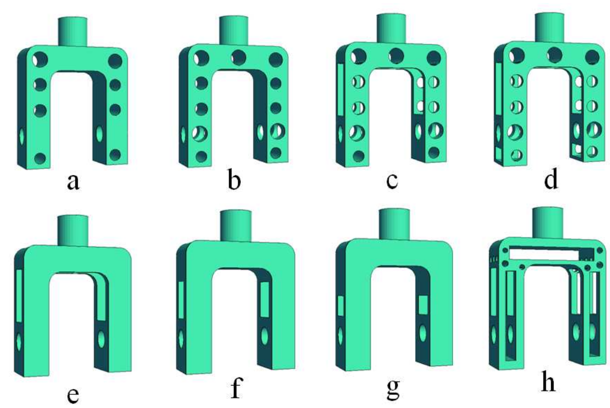



Figure 22: EIGHT INITIAL DESIGNS FOR THREE-HOLES BRACKET.

For the initial designs in Figure 22, the estimated $\Delta J/J_o$ versus actual $\Delta J/J_o$ is illustrated in Figure 23. Observe that: (1) the solid line represents an ideal scenario where estimation coincides with actual, and (2) there is a good correlation between the estimated and actual metrics. Based on Figure 22, one can conclude that design 'h' is far from optimal, while design 'a' is close to optimal.

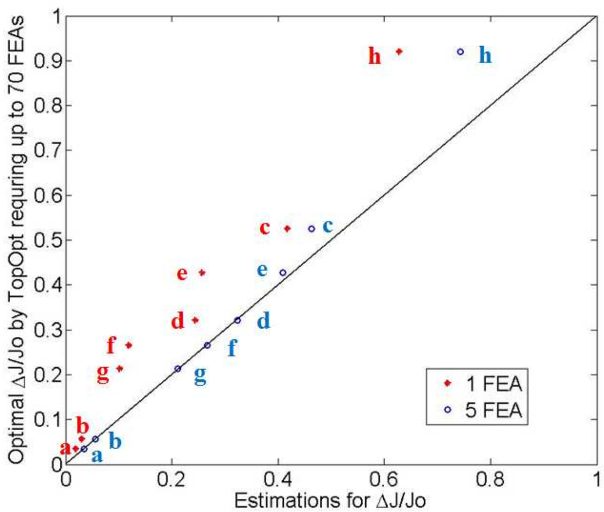



Figure 23: ESTIMATION FOR POTENTIAL COMPLIANCE IMPROVEMENT FOR INITIAL DESIGNS IN Figure 22.

Similarly, Figure 24 illustrates the predicted metrics $\Delta V/Vo$ using 1 and 5 FEAs. Once again, design ‘h’ is furthest from being optimal while design ‘a’ is closest to being optimal.

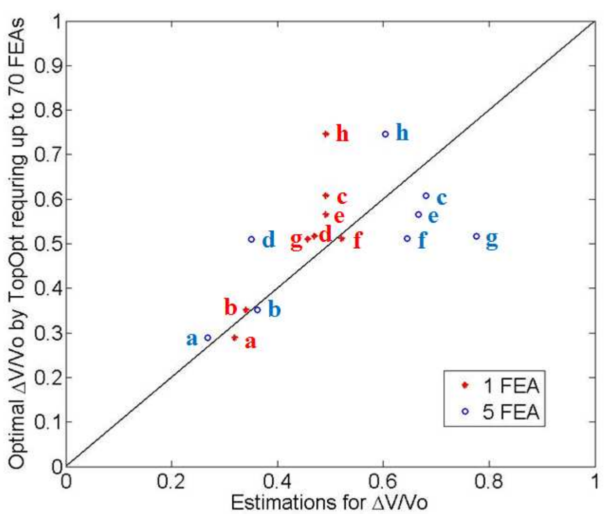



Figure 24: ESTIMATION FOR POTENTIAL VOLUME REDUCTION FOR INITIAL DESIGNS IN Figure 22.

If we combine the two estimations (Figure 23 and Figure 24), one can arrive at Figure 25, that illustrates the benefits with respect to two different criteria.

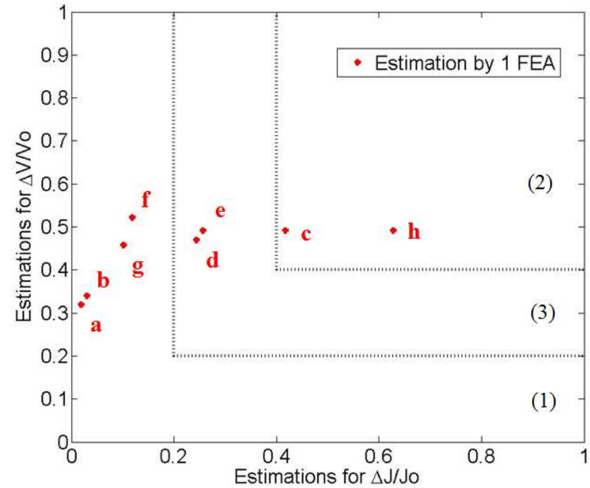
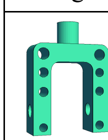
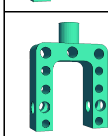
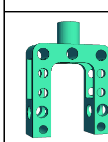


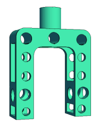
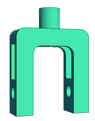
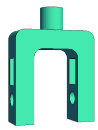
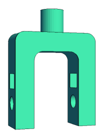
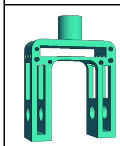

Figure 25: ESTIMATION FOR POTENTIALS OF COMPLIANCE IMPROVEMENT AND VOLUME REDUCTION FOR INITIAL DESIGNS BY 1 FEA IN Figure 22.

In Figure 25, based on the distances that design points are from the two axis, we propose to divide the region into three zones as shown:

1. If a design falls in the 0%-20% zone, it close to being optimal; designs “a, b, g, and f” fall into this category.
2. If a design falls in the 40%-100% zone, the potential for further optimization is significant; designs “c and h” fall into this category.
3. Finally, if a design falls in 20%-40% (“fuzzy zone”), the designer has two options: (a) if computing resource is limited, then do *not* optimize; (b) if computing resources are available, then carry out a 5-iteration-FEA based estimation for more accurate estimation; designs “e and d” fall into this category.

Table 3: OPTIMIZATION SUGGESTIONS FOR INITIAL DESIGNS IN Figure 22.

| Initial designs | Optimize? | Optimization rank based on 1-FEA |
|---|---|---|
|  | NO | 8 |
|  | NO | 7 |
|  | YES | 2 |

| | | |
|---|---|---|
|  | Unsure | 4 |
|  | Unsure | 3 |
|  | NO | 5 |
|  | NO | 6 |
|  | YES | 1 |

## CONCLUSIONS AND FUTURE WORK

The main question raised in this paper is: "*Can one predict the benefits of topology optimization?*"

Based on the case-studies, we believe the answer is a cautious "Yes". In particular, using a few (~5) finite element studies, we showed that one can predict the benefits of topology optimization (for a given scenario).

The present work focused only on compliance and volume fraction. We believe it can be extended to other quantities of interest, e.g. von Mises stress and eigen-values, since the concept of topological sensitivity can be extended to such quantities as well [44], [49], [12]. Future work will also focus on assembly of parts (see Figure 4). Specifically, which of the parts, if any, should one optimize? How do we rank-order these parts for optimization?

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